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Flows with variable viscosity: an asymptotic model

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Abstract. Introducing a *variable* dynamic viscosity coefficient in the Navier equations for an incompressible fluid of small viscosity ($\mathrm{Re} \gg 1$, where Re is the classical Reynolds number), we exhibit a *three-layer* asymptotic model: *ideal* fluid layer, *boundary* layer and *lower viscous* layer. Surprisingly we find that the interaction between the boundary layer and the lower viscous layer is realized only starting with the *second-order* approximation. We give the full mathematical formulation of the corresponding boundary-layer problem, for the second approximation, with the new boundary conditions obtained by matching from the first-order lower viscous layer, of which the thickness is $O(1/\mathrm{Re})$.

As an application of this three-layer asymptotic model we solve completely the classical Blasius problem. In this case the expression for the skin friction coefficient shows that the classical Blasius value is multiplied by a positive term, directly linked to the variability of the dynamic viscosity coefficient.

1. Introduction

We consider steady, two-dimensional flow past a realistic body of nonvanishing thickness. For definiteness we may envisage an airfoil spanning the test section of a wind tunnel with plane walls, so that the flow would be uniform in the absence of the airfoil. In any case the body is assumed to be solid, with an impermeable surface. Its shape will define a fixed reference length (L_0), and hence a Reynolds number (Re) based on that, so that a nondimensional formulation is possible from the start. Let $\mathbf{u}$ be the dimensionless velocity and p the dimensionless pressure.

A rational approach is to assume that, for a fluid of small viscosity, the flow is one differing appreciably from that of an ideal fluid only in the vicinity of the body surface. For that reason it is convenient to use curvilinear, orthogonal coordinates, usually denoted by s and n, such that the body surface is the line $n = 0$. A regular system of such coordinates certainly exists, in any case in a sufficiently small neighborhood of the body surface, provided that the body surface has no corner. A formulation of Navier equations in such coordinates is found in Goldstein ([1], p. 119). It is convenient to measure s along the body surface from the stagnation point that must be anticipated near the rounded nose of a realistic body, and to begin with it is desirable to exclude a neighborhood of the point $s = n = 0$ from consideration. It is similarly desirable to exclude a neighborhood of the tail of the body.

Since n measures normal distance from the body surface, we write

$$\mathbf{u} = u\boldsymbol{\tau} + v\boldsymbol{\nu} , \qquad \nabla = h(s) \frac{\partial}{\partial s} \boldsymbol{\tau} + \frac{\partial}{\partial n} \boldsymbol{\nu} , \tag{1}$$

with $h(s) = [1 + K(s)n]^{-1}$, where $K(s)$ is the curvature of the body surface; $K(s)$ and its first derivative $\mathrm{d}K/\mathrm{d}s$ are bounded (or, in any case, $K(s)/\mathrm{Re}^{1/2} \to 0$ and $\mathrm{Re}^{-1}\,\mathrm{d}K/\mathrm{d}s \to 0$ as $\mathrm{Re} \to \infty$).

We start from the Navier equations in dimensionless form:

$$\nabla \cdot \mathbf{u} = 0\,,$$
$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \varepsilon^2 \left\{ \mu(n)\nabla^2 \mathbf{u} + \frac{\mathrm{d}\mu}{\mathrm{d}n}\left[\frac{\partial \mathbf{u}}{\partial n} + \nabla v\right]\right\}, \tag{2}$$

where†

$$\mu(n) = 1 + \tilde{\mu}(n/\Delta(\varepsilon))\,, \tag{3}$$

with $\tilde{\mu}(\infty) = 0$, $\Delta(\varepsilon) \ll \varepsilon$ and $\varepsilon = \mathrm{Re}^{-1/2}$. We shall see that $\Delta(\varepsilon) = \varepsilon^2$ and the thickness of the lower viscous layer is $\mathrm{O}(\mathrm{Re}^{-1})$. The case where $\Delta(\varepsilon) \gg \varepsilon$ is more subtle and it requires the application of the homogenization technique for the microscopic description related to $\tilde{\mu}(n/\Delta(\varepsilon))$.

2. The associated three limiting processes

The *first* limiting process is the usual limit of the exact solution of (2), with (3) and boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } n = 0 \tag{4}$$

as $\varepsilon \to 0$ for *fixed* s and n. In this case we search for an asymptotic representation of the flow:

$$u = \bar{u}_0 + \varepsilon \bar{u}_1 + \cdots\,, \qquad v = \bar{v}_0 + \varepsilon \bar{v}_1 + \cdots\,, \qquad p = \bar{p}_0 + \varepsilon \bar{p}_1 + \cdots\,. \tag{5}$$

As usual, we get to leading order the classical Euler equations for $\bar{u}_0$, $\bar{v}_0$ and $\bar{p}_0$, while to next order for $\bar{u}_1$, $\bar{v}_1$ and $\bar{p}_1$ we have the linearized perturbation equations relative to the Euler-preceding equations (see, for instance, Zeytounian [4], p. 40).

The *second* limiting process is the inner limit of the exact solution as a function of s and $\hat{n} = n/\varepsilon$, as $\varepsilon \to 0$ for fixed s and $\hat{n}$. In this case, instead of (5), write:

$$u = \hat{u}_0 + \varepsilon \hat{u}_1 + \cdots\,, \qquad v = \varepsilon \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \cdots\,, \qquad p = \hat{p}_0 + \varepsilon \hat{p}_1 + \cdots\,. \tag{6}$$

For $\hat{u}_0$ and $\hat{v}_1$ we have the classical Prandtl boundary-layer equations and the following relations are obtained:

† With dimensions we have: $\tilde{\mu}^* = \mu_0 \tilde{\mu}(n^*/l_0)$, where l_0 is a characteristic length for the *microscopic* variation of the dynamic viscosity coefficient. If L_0 is the reference length associated with the shape of the body, then $\Delta(\varepsilon) = l_0/L_0 \ll \varepsilon \ll 1$.

$$\lim_{n\to 0} \bar{v}_0 = 0\,, \qquad \lim_{\hat{n}\to\infty} \hat{u}_0 = \bar{u}_0(s,0) = u_{e0}(s)\,,$$
$$\hat{p}_0 = \bar{p}_0(s,0) = p_{0e}(s)\,, \qquad \frac{\mathrm{d}p_{0e}}{\mathrm{d}s} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(u_{e0}^2) = 0\,, \qquad \bar{v}_1(s,0) = \int_0^\infty \left[\frac{\mathrm{d}u_{e0}}{\mathrm{d}s} - \frac{\partial \hat{u}}{\partial s}\right]\mathrm{d}\hat{n} \tag{7}$$

after the matching of (6) with (5).

For $\hat{u}_1$ and $\hat{v}_2$ we obtain the boundary-layer equations of the *second* approximation, according to Van Dyke [3], where we have curvature effects,

$$\frac{\partial \hat{u}_1}{\partial s} + \frac{\partial \hat{v}_2}{\partial \hat{n}} = -K(s)\frac{\partial}{\partial \hat{n}}(\hat{n}\hat{v}_1)\,,$$
$$\frac{\partial}{\partial s}(\hat{u}_0\hat{u}_1) + \hat{v}_2\frac{\partial \hat{u}_0}{\partial \hat{n}} + \hat{v}_1\frac{\partial \hat{u}_1}{\partial \hat{n}} + \frac{\partial \hat{p}_1}{\partial s} - \frac{\partial^2 \hat{u}_1}{\partial \hat{n}^2} = K(s)\left\{\hat{n}\hat{u}_0\frac{\partial \hat{u}_0}{\partial s} - \hat{u}_0\hat{v}_1 + \hat{n}\frac{\mathrm{d}p_{e0}}{\mathrm{d}s} + \frac{\partial \hat{u}_0}{\partial \hat{n}}\right\}, \tag{8}$$

and matching gives

$$\lim_{\hat{n}\to\infty} \hat{u}_1 = \bar{u}_1(s,0) = u_{e1}(s)\,, \qquad \hat{p}_1 = \bar{p}_1(s,0) + K(s)\int_{\hat{n}}^{\infty} \hat{u}_0^2\,\mathrm{d}\hat{n}\,. \tag{9}$$

For the time being we don't have the possibility to take into account the influence of the microscopic coefficient $\tilde{\mu}(n/\Delta)$ and for that reason we envisage a tentative *third* limiting process: $\varepsilon\to 0$ for fixed s and $\tilde{n} = n/\Delta$, with the following asymptotic representation of the flow:

$$u = \varepsilon^{\alpha}\tilde{u}_0(s,\tilde{n}) + \cdots\,, \qquad v = \varepsilon^{\beta}\tilde{v}_0(s,\tilde{n}) + \cdots\,, \qquad p = \tilde{p}_0(s,\tilde{n}) + \varepsilon\tilde{p}_1(s,\tilde{n}) + \cdots\,, \tag{10}$$

with constants α and β.

If we assume that $\Delta(\varepsilon) = \varepsilon^{\lambda}$, $\lambda > 1$, then for $\lambda = \beta - \alpha$ we obtain for $\tilde{u}_0$, $\tilde{v}_0$ and $\tilde{p}_0$ the following system of equations in the *lower* viscous layer:

$$\frac{\partial \tilde{u}_0}{\partial s} + \frac{\partial \tilde{v}_0}{\partial \tilde{n}} = 0\,, \qquad \tilde{p}_0 = p_{0e}(s)\,,$$
$$\frac{\partial}{\partial \tilde{n}}\left\{[1+\tilde{\mu}(\tilde{n})]\frac{\partial \tilde{u}_0}{\partial \tilde{n}}\right\} = 0\,, \tag{11}$$

and we have, according to (4), the conditions

$$\tilde{u}_0(s,0) = \tilde{v}_0(s,0) = 0\,. \tag{12}$$

3. Interaction between the boundary layer and the lower viscous layer

The solution of the lower viscous layer problem, (11), (12), is easy, and particularly we get

$$\tilde{u}_0(s,\tilde{n}) = A_0(s)\int_0^{\tilde{n}} (1+\tilde{\mu}(t))^{-1}\,\mathrm{d}t\,. \tag{13}$$

Now it is necessary to elucidate the behavior of (13) as $\tilde{n}\to\infty$. Through a straightforward argument we obtain

$$\tilde{u}_0(s, \tilde{n}) \sim A_0(s)\{\tilde{n} + \tilde{U}_0^\infty + \cdots\} \quad \text{as } \tilde{n} \to \infty\,, \tag{14}$$

where

$$\tilde{U}_0^\infty = \int_0^\infty \{-1 + (1 + \tilde{\mu}(t))^{-1}\}\, \mathrm{d}t < \infty\,, \tag{15}$$

and we see that this imposes a constraint on $\tilde{\mu}(\tilde{n})$. But, according to (6), in the boundary layer we have also the following behavior for u:

$$u \sim \hat{u}_0(s, 0) + \hat{n}\, \frac{\partial \hat{u}_0}{\partial \hat{n}}\bigg|_{\hat{n}=0} + \varepsilon \hat{u}_1(s, 0) + \cdots \quad \text{as } \hat{n} \to 0\,. \tag{16}$$

Therefore, taking into account (10) and the relation $\hat{n} = \varepsilon^{\lambda-1}\tilde{n}$, we verify that the matching between (10) and (6) is possible, according to (14) and (16), if

$$\alpha = 1\,, \qquad \lambda = \alpha + 1 = 2 \quad \text{and} \quad \beta = 3\,, \tag{17}$$

and in this case we obtain the following relations

$$\begin{aligned} &\hat{u}_0(s, 0) = 0\,, \qquad A_0(s) = \frac{\partial \hat{u}_0}{\partial \hat{n}}\bigg|_{\hat{n}=0}\,, \\ &\hat{u}_1(s, 0) = A_0(s)\, \tilde{U}_0^\infty\,, \qquad \hat{v}_1(s, 0) = 0\,, \\ &\hat{v}_2(s, 0) = 0\,. \end{aligned} \tag{18}$$

From (18) we see that the classical laminar boundary-layer problem, for $\hat{u}_0$ and $\hat{v}_1$, is not affected by the appearance of the lower viscous layer. This leading-order lower viscous layer is *active* at the level of equations (8) for $\hat{u}_1$ and $\hat{v}_2$, with (9), in such a way that the boundary conditions on $\hat{n} = 0$ for equations (8) are

$$\hat{u}_1(s, 0) = A_0(s)\tilde{U}_0^\infty\,, \qquad \hat{v}_2(s, 0) = 0\,, \tag{19}$$

where $A_0(s)$ is defined by the second relation of (18), while $\tilde{U}_0^\infty$ is a constant according to (15).

As $\Delta(\varepsilon) = \varepsilon^\lambda = \varepsilon^2 = \mathrm{Re}^{-1}$, the leading-order lower viscous layer, governed by the equations (11), is a thin layer (within the boundary layer) with thickness $\mathrm{O}(\mathrm{Re}^{-1})$. But for the time being we don't know if the boundary-layer problem (8) with (9) and (19) is well posed or not. It seems that the proof of uniqueness of a solution in $C^\infty(0, \infty)$ is not easy! Therefore, in the following section, we consider the particular case of the classical Blasius problem.

4. Application to the Blasius problem*

In this case we have for the stream function $\psi(x, y)$, such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, the following three asymptotic representations:

* A basic problem of the theory of fluids of small viscosity is that of steady flow past a solid flat plate placed edgewise in a uniform stream. More precisely, the plate is understood to be a half-plane, say $y = 0$, $0 < x < \infty$ (see Meyer [2], Chapter 4).

$$\psi = y - \varepsilon\beta \,\text{Real}\,(\sqrt{x+\text{i}y}) + \cdots ,$$
$$\psi = \varepsilon\sqrt{x}\, f_0(\eta) + \varepsilon^2 f_1(\eta) + \cdots , \tag{20}$$
$$\psi = \varepsilon^3 x^{-1/2} f_0''(0) F(\tilde{y}) + \cdots ,$$

where

$$\beta \simeq 1.7208 , \qquad \hat{y} = y/\varepsilon , \qquad \tilde{y} = y/\varepsilon^2 , \qquad \eta = x^{-1/2}\hat{y} . \tag{21}$$

The function $f_0(\eta)$ must satisfy the classical Blasius equation and for $F(\tilde{y})$ we have the following relation:

$$F(\tilde{y}) = \int_0^{\tilde{y}} \left\{ \int_0^{\alpha} (1 + \tilde{\mu}(t))^{-1} \,\mathrm{d}t \right\} \mathrm{d}\alpha . \tag{22}$$

Finally, we obtain the following linear boundary-value problem for $f_1(\eta)$:

$$2f_1''' + f_0 f_1'' + f_0' f_1' = 0 ,$$
$$f_1(0) = 0 , \qquad f_1'(0) = \alpha_0 f_0''(0) , \qquad f_1'(\infty) = 0 , \tag{23}$$

with*

$$\alpha_0 = -\int_0^{\infty} \tilde{\mu}(t)(1 + \tilde{\mu}(t))^{-1} \,\mathrm{d}t .$$

The solution of (23) is easy and we find that

$$f_1(\eta) = \alpha_0 f_0'(\eta) + \text{Constant} \int_0^{\eta} \left\{ f_0''(t) \int_0^{t} (f_0''(u))^{-1} \,\mathrm{d}u \right\} \mathrm{d}t . \tag{24}$$

Unfortunately we do not have the possibility to determine the constant in (24), since the condition at infinity, $f_1'(\infty) = 0$, is automatically satisfied. Then for the determination of the "Constant" in the solution (24) we have to consider the matching between the first and the second representation of (20) and we find the following *complementary condition*:**

$$f_1''(\infty) = 0 . \tag{25}$$

With (25), the solution of (23), according to (24), is simply

$$f_1(\eta) = \alpha_0 f_0'(\eta) . \tag{26}$$

Now we are able to write a uniform asymptotic representation in the vicinity of $y = 0$ for the horizontal velocity,

* We note that $A_0(x) = f_0''(0)x^{-1/2}$ according to the second relation of (18) and the second representation of (20).
** For the Blasius problem the tables of $f_0''(\eta)$ show that $f_0''(\infty) = 0$ and this last condition results from the matching with the uniform flow.

$$u = \partial\psi/\partial y = f_0'(y/\varepsilon\sqrt{x}) - \varepsilon f_1'(0)/\sqrt{x} + \varepsilon[f_0''(0)F'(y/\varepsilon^2) + f_1'(y/\varepsilon\sqrt{x})]/\sqrt{x} + O(\varepsilon^2)\,. \tag{27}$$

Therefore, we obtain for the skin friction coefficient

$$C_f = 2f_0''(0)\,\mathrm{Re}_{x^*}^{-1/2}[1 + (1 + \tilde{\mu}(0))^{-1}] \tag{28}$$

where Re_{x^*} is the relevant local Reynolds number for the flat plate ($x^* = L_0 x$ is a dimensional variable). In the particular case when $\tilde{\mu}(t) = \exp(-\omega t)$, $\omega > 0$, we derive for C_f, according to (28), the following expression,

$$C_f = 3\,\mathrm{Re}_{x^*}^{-1/2} f_0''(0)\,. \tag{29}$$

5. Conclusion

If we take into account the complementary condition (25), derived for the Blasius problem in Section 4, then it seems that the uniqueness of a solution of the linear boundary-layer problem (8), (9), (19) is possible if we suppose that this last solution satisfies also the following behavior at infinity:*

$$\lim_{\hat{n}\to\infty} \frac{\partial \hat{u}_1}{\partial \hat{n}} = \lim_{n\to 0} \frac{\partial \bar{u}_0}{\partial n}\,, \tag{30}$$

which follows from the matching between (5) and (6) for $\partial u/\partial n$.

References

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* From the classical Prandtl boundary-layer equations, for $\hat{u}_0$ and $\hat{v}_1$, we can see that: *if the relations (7) are really satisfied, then* $\partial\hat{u}_0/\partial\hat{n}$ *and* $\partial^2\hat{u}_0/\partial\hat{n}^2 \to 0$ as $\hat{n}\to\infty$.